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# An Infinite Number of Effectively Infinite Clusters in Critical Percolation

**Naeem Jan,[1] Dietrich Stauffer,[2,3] and Amnon Aharony[2]**

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An infinite number of effectively infinite clusters are predicted at the percolation threshold, if "effectively infinite" means that a cluster's mass increases with a positive power of the lattice size $L$. All these cluster masses increase as $L^D$ with the fractal dimension $D = d - \beta/\nu$, while the mass of the $r$th largest cluster for fixed $L$ decreases as $1/r^\lambda$, with $\lambda = D/d$ in $d$ dimensions. These predictions are confirmed by computer simulations for the square lattice, where $D = 91/48$ and $\lambda = 91/96$.

**KEY WORDS:** Percolation; ranking; size distribution; infinite clusters.

For percolation, every site of a large lattice of linear size $L$ in $d$ dimensions is randomly occupied with probability $p$, and clusters are sets of neighbouring occupied sites. We call $s$ the mass of the cluster and define clusters as effectively infinite if their (average) mass goes to infinity with a positive power of $L$ as $L \to \infty$. For $p$ below the percolation threshold $p_c$, the mass of the largest cluster goes to infinity for $L$ tending to infinity, but only logarithmically.(1) These clusters are therefore not "effectively infinite" by our definition. At the percolation threshold, the mass of the largest cluster scales as $L^D$, with the fractal dimension $D = 91/48$ in $d = 2$ dimensions.(2) Thus this mass diverges to infinity as a positive power of $L$ for $L \to \infty$. The largest cluster is called "the incipient infinite cluster" and is effectively infinite according to our definition. At $p_c$, the number (per site) $n_s$ of

[1] Physics Department, St. Francis Xavier University, Antogonish, Box 5000, Nova Scotia, Canada B2G 2W5.
[2] School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel.
[3] Institute for Theoretical Physics, Cologne University, 50923 Cologne, Germany.



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clusters containing $s$ sites each decays as $s^{-\tau}$ and $\tau = 1 + d/D = 187/91$ for $d = 2$, as recently reconfirmed[3] using a square lattice with $L = 2$ million, the largest lattices (to our knowledge) ever simulated. The present note looks at other large clusters, which are not the largest but nevertheless become effectively infinite in the above sense.

Some mathematical publications proved that what they called "the infinite cluster" is unique.[4] However, their definition of "infinite" was not the same as our definition of "effectively infinite," and it did not apply to the extrapolation of finite samples to infinity, as must be done in all computer simulations, including the present problem. Furthermore, that definition also does not apply to *spanning* clusters, which were shown to be non-unique.[5,6] We have searched the mathematical physics literature, and failed to find a clean definition of an "infinite cluster" which would help us in a numerical test on extrapolated finite samples. This is the reason for our new definition.

Already a long time ago,[7] the masses of the second and third-largest cluster were investigated by simulations at the threshold, and were found to be proportional to that of the largest one. This leads to the conclusion that the second largest cluster also scales as $L^D$. This non-uniqueness of the

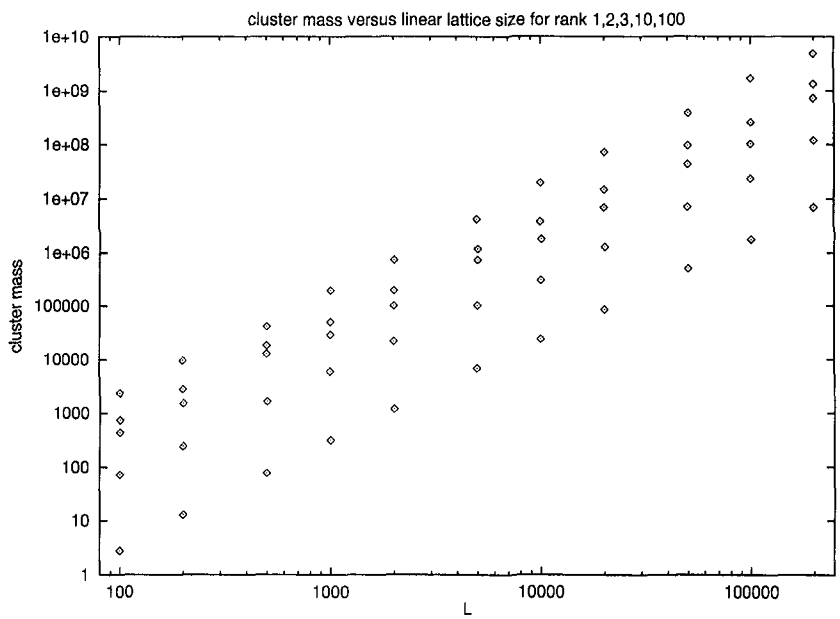


Fig. 1. Cluster mass versus lattice size $L$ for rank 1, 2, 3, 10, and 100 (from top to bottom). Typically we averaged here and in the later figures over 20 samples for each $L$.

effectively infinite cluster is investigated here more systematically. In particular, we present a systematic study of the dependence of the $r$th largest cluster on its "rank" $r$. In an earlier effort of Watanabe,[8] the dependence of the cluster mass on its rank was wrongly fitted to what the author called "Zipf's law," $s_r \propto 1/r$. This was done in both two and three dimensions. As we will explain here, there is no reason to expect such a law, and the fit simply resulted from the small samples, from the closeness of the exponent to 1 and from using data for $p < p_c$. We note that Zipf's law is not as general as some authors imply!

At $p_c$, $n_s$ is proportional to $s^{-\tau}$, and therefore the total number of clusters with mass $> s$ in a sample of linear size $L$ varies as $N_s \propto L^d s^{1-\tau}$. As a result,[9] the average mass of the $r$th largest cluster should be

$$s_r \propto L^D / r^{\lambda}, \qquad (1)$$

where $\lambda = 1/(\tau - 1) = D/d = 91/96$ in $d = 2$ and where we have used the expression $\tau = 1 + d/D = 187/91$. Thus, the masses of all the large clusters are proportional to that of the largest one, with an amplitude which decreases with the cluster's rank $r$.

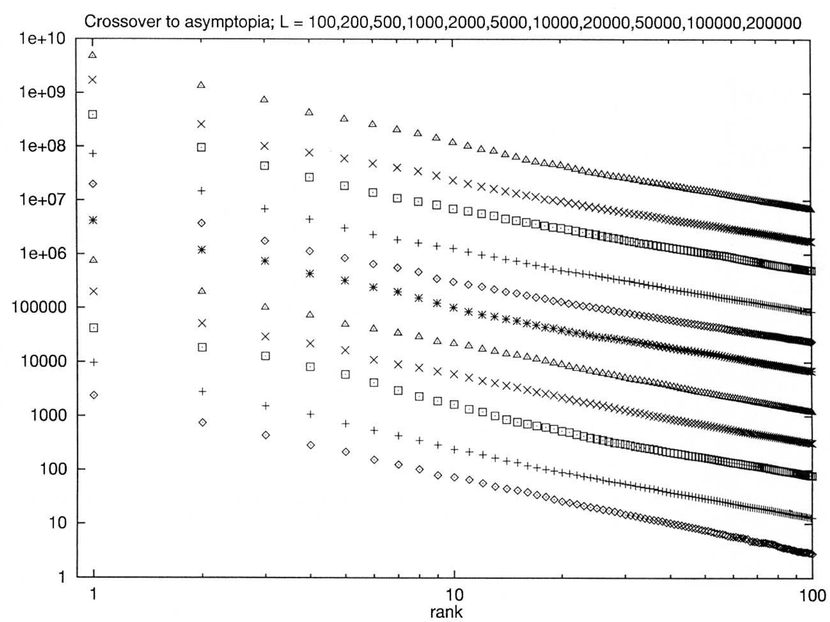


Fig. 2. Cluster mass versus rank for lattice size $L/100 = 1$, 2, 5, 10, 20, 50, 100, 200, 500, 1000, and 2000 (from bottom to top).

More precisely, in a large but finite lattice, finite-size scaling predicts at $p = p_c$:[2]

$$N_s = L^d s^{1-\tau} f(s/L^D) \tag{2}$$

with the scaling function $f(z)$ approaching a constant for small $z = s/L^D$ and going to zero for large $z$. Thus for the largest clusters some deviations are expected and seen.[3] However, Eq. (2) still predicts that the rank $r$ is a function of the scaled variable $z_r = s_r/L^D$, $r = z_r^{1-\tau} f(z_r)$, approaching the power law $z_r^{1-\tau}$ for small $z_r$, or equivalently for large $r$. Therefore, Eq. (1) is not exact for the largest or second largest cluster, but it becomes accurate for large ranks $r$. Eq. (2) is also not exact for small clusters, $s = 1, 2,...,$ when $N_s \propto L^d$. We conclude that Eq. (1) is expected to be correct for the intermediate range $1 \ll r \ll L^d$.

We tested these predictions using the Hoshen-Kopelmen routine with the Nakanishi method of recycling redundant labels,[10] for the square $L \times L$ lattice at $p_c = 0.592746$. The Hoshen–Kopelman method is used here

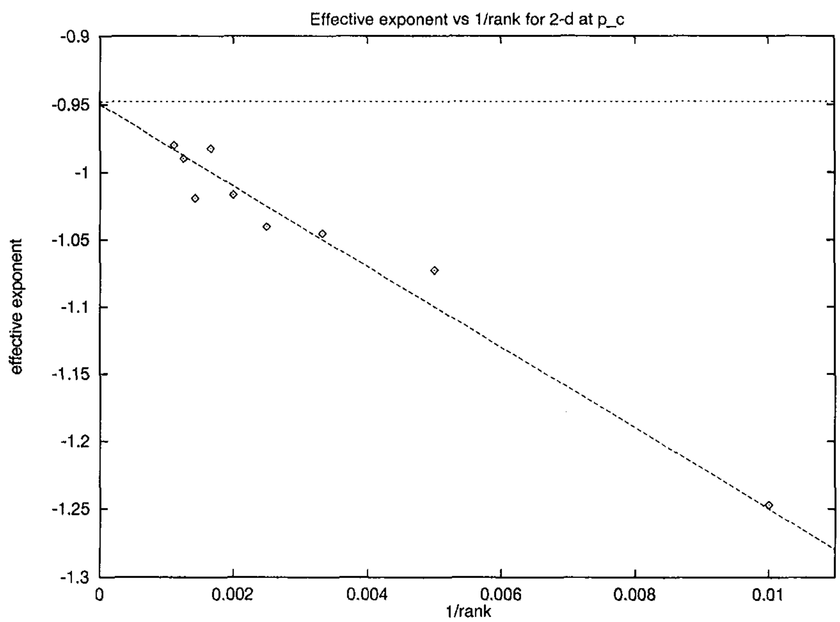


Fig. 3. Effective exponent $\lambda_{\text{eff}}$ of cluster mass versus inverse rank $r$, for $10 < r < 1000$ and $L = 200,000$. We have subdivided the data into ten regions of roughly equal size on a logarithmic scale in the rank. For each region we find by a least-squares fit the best slope in log(mass) versus log(rank). This value of $\lambda_{\text{eff}}$ is shown with the average rank for the region. This exponent should converge towards 91/96 for $r \to \infty$.

for two-dimensional random site percolation in the following manner. We consider one row at a time and randomly determine if a site is occupied or vacant. If vacant we label the site as vacant and move to the next site. If occupied we look at the neighbour to the left and top. If these are both empty then we assign a new label and a size of unity to this occupied site. If both neighbours are occupied, or one is occupied, then we assign the lowest label to the new site and we also adjust the size of this growing cluster. All other labels from the neighbours are redundant. The Nakanishi recycling of labels is used to compress the labels by removing all redundant labels from the growing percolation system.

Figure 1 shows the average masses for $r = 1, 2, 3, 10$ and 100 as a function of $L$; we see nearly parallel straight lines with the expected slope $D = 1.9$ in this log-log plot (see ref. 2 for better data for this fractal dimension). Figure 2 shows for various $L$ the mass of the 100 largest clusters as a function of rank $r$; in the right part of the figure we see parallel straight lines with a slope near unity. More precisely, Fig. 3 shows for $L = 200{,}000$ and rank $r$ up to 1000 the effective exponent $\lambda$ (negative slope of the log-log plot in Fig. 2) versus the reciprocal rank; the data can be extrapolated to an asymptotic exponent slightly smaller than one, roughly compatible with the theoretical prediction $\lambda = 91/96 \simeq 0.95$ of Eq. (1).

Thus again scaling theory is confirmed, and at the threshold already in two dimensions we have infinitely many effectively infinite clusters.

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